Exploration of the topology of offsets of a plane curve: the graphical-algebraic interplay between automated methods of a DGS and the algebraic power of a CAS.

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Offsets of plane curves is an ancient topic (Euler called them parallel curves).

Let $\mathcal{C}$ be a plane curve and d a positive real number. At a regular point A on $\mathcal{C}$, define a unit normal vector $\overrightarrow{N_A}$. The points B and C are defined by $\overrightarrow{AB} = d \cdot \overrightarrow{N_A}$ and $\overrightarrow{AC} = -d \cdot \overrightarrow{N_A}$. The geometric loci of the points B and C when A runs over $\mathcal{C}$ are called the offset of $\mathcal{C}$ at distance d. We call one of them *outer offset* and the other one *inner offset*, the distinction is quite arbitrary, as the intuition may change when d varies. The curve $\mathcal{C}$ is called the progenitor of the offset.

The topology of an offset may be much more complicated than the topology of the progenitor. If $\mathcal{C}$ is an ellipse and d small enough, both components are non-singular. For larger values of d, the outer offset is still non-singular, but the inner offset may present cusps. For progenitors of higher degree, an offset may present cusps and points of self-intersection (crunodes). The first determination of an offset may provide a parametric representation. By successive differentiation, it is possible to determine singular points and their nature. This will show the cusps, but computation machinery may be heavy and, even when possible, time consuming.

Implicitization, provided either by a specific command or using Maple's *PolynomialIdeals* package, yields a polynomial equation, also sometimes a heavy process. Polynomial factorization answers the question whether the offset if irreducible or not, but the components may not be distinguished by algebraic means. Irrelevant components may appear, for topological reasons related to Zariski topology. This distinction can be obtained during an exploration with Dynamic Geometry Software: GeoGebra-Discovery offers automated methods. Fig. 2 shows the graphical output (notre the different topologies for different distances), but a symbolic equation is not provided and strong algebraic computations are needed.

Using the symbolic equation, we have another method to determine the cusps of the offset; the algebraic machinery is now based on the curvature.

In our talk, we show the code developed with Maple for the above explorations. We exemplify it for offsets of ellipses (Fig. 1), Kiss curves (Fig. 3) and other plane curves, showing that the methods can be applied generally. The power of the specific computer in use fixes the constraints on the algebraic manipulations.

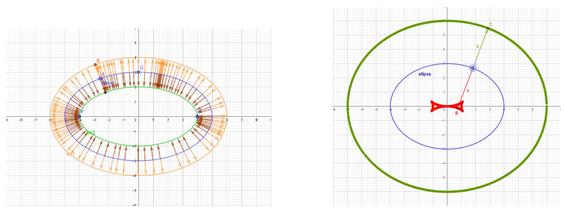

Figure 1: Examples of offsets of ellipses

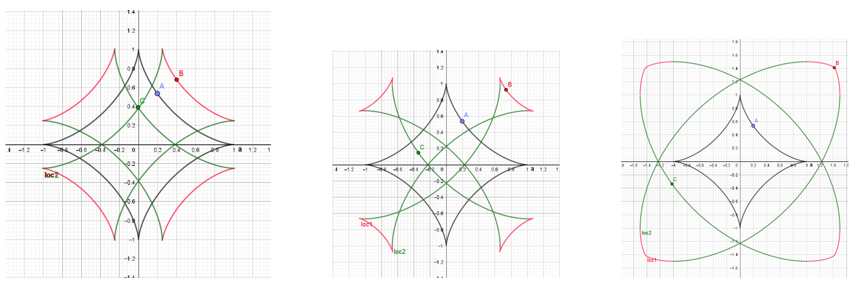

Figure 2: Examples of offsets of an astroid

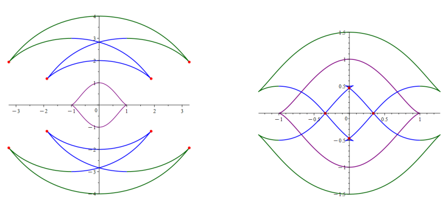

Figure 3: Examples of offsets of a kiss curve - cusps and self-intersections are emphasized