Impulse and Momentum

All particles with mass experience the effects of impulse and momentum. Momentum and inertia are similar concepts that describe an objects motion, however inertia describes an objects resistance to change in its velocity, and momentum refers to the magnitude and direction of it's motion. Momentum is an important parameter to consider in many situations such as braking in a car or playing a game of billiards. An object can experience both linear momentum and angular momentum. The nature of linear momentum will be explored in this module.

This section will discuss momentum and impulse and the interconnection between them. We will explore how energy lost in an impact is accounted for and the relationship of momentum to collisions between two bodies. This section aims to provide a better understanding of the fundamental concept of momentum.

▼ Understanding Momentum

Any body that is in motion has momentum. A force acting on a body will change its momentum. The momentum of a particle is defined as the product of the mass multiplied by the velocity of the motion. Let the variable L represent momentum.

$$L = m \cdot v$$

... Eq. (1)

▼ The Principle of Momentum

Recall Newton's second law of motion.

$$F = m \cdot a$$

... Eq. (2)

This can be rewritten with accelleration as the derivate of velocity with respect to time.

$$F = m \cdot \frac{d}{dt} v$$

... Eq. (3)

If this is integrated from time t_1 to t_2

$$\int_{t_1}^{t_2} F \, dt = m \cdot v_2 - m \cdot v_1$$

... Eq. (4)

Moving the initial momentum to the other side of the equation yields

$$m \cdot v_1 + \int_{t_1}^{t_2} F \, dt = m \cdot v_2$$

... Eq. (5)

Here, the integral in the equation is the impulse of the system; it is the force acting on the mass over a period of time t_1 to t_2.

If a force acts on a particle for a short period of time it is called an impulsive force. If this force causes a definite change in momentum, the resulting motion is called impulsive motion. In the case of several particles interacting, Eq. (5) can be rewritten as the sum of the momenta and impulses.

$$\sum m \cdot v_1 + \sum F \cdot \Delta t = \sum m \cdot v_2$$

<div align="right">... Eq. (6)</div>

▼ Collisions

When two particles collide and there are no external impulsive forces acting on them, total momentum is conserved. The equation becomes

$$\sum m \cdot v_0 = \sum m \cdot v_1$$

<div align="right">... Eq. (7)</div>

If two particles of mass m_a and m_b collide, their resulting velocities will change. The motion of these particles can be expressed by v_0 and v_1 before and after the collision, respectively, in the following equation.

$$m_a v_{a0} + m_b v_{b0} = m_a v_{a1} + m_b v_{b1}$$

<div align="right">... Eq. (8)</div>

▼ Line of Impact

The direction at which two objects hit each other will determine the trajectory of their path. The line of impact is the common normal line between the contacting areas of the particles during impact.
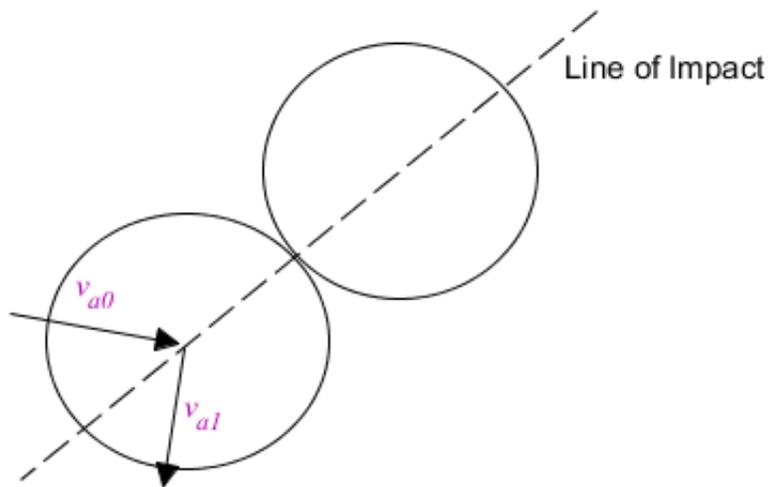
Figure 1: Line of impact of two particles

A **central impact** is an impact of which both particles have their center of mass on the line of impact. An example of two equal masses hitting each other is shown below.

Figure 2: Two equal mass balls under direct central impact

The momentum of the black ball is moved to the red ball after impact. Here, the balls are moving and hitting each other on the line of impact, so the second ball will travel in the same direction as the first. This is called direct central impact. Some particles

may hit each other off-center (or off the line of impact) which will cause them to move in different directions, called oblique central impact.
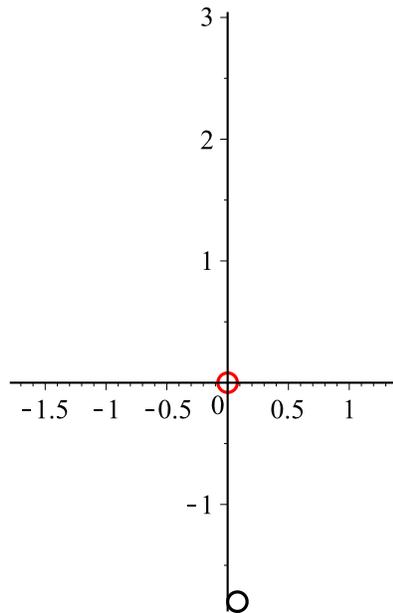
Figure 3: Two equal mass balls under oblique central impact

If the particles are not travelling in the same direction before and after the impact then the components of each particle must be considered. The conservation of momentum still applies if the vector is separated. In a general *xyz* space this would imply

$$m_a \cdot v_{ax0} + m_b \cdot v_{bx0} = m_a \cdot v_{ax1} + m_b \cdot v_{bx1} \quad \text{and} \quad m_a \cdot v_{ay0} + m_b \cdot v_{by0} = m_a \cdot v_{ay1} + m_b \cdot v_{by1}$$

$$\text{and} \quad m_a \cdot v_{az0} + m_b \cdot v_{bz0} = m_a \cdot v_{az1} + m_b \cdot v_{bz1}$$

... Eq. (9.1-9.3)

which can be simplified and represented by a vector

$$m_a \vec{v}_{a0} + m_b \vec{v}_{b0} = m_a \vec{v}_{a1} + m_b \vec{v}_{b1}$$

... Eq. (10)

Elastic and Inelastic Collisions

It is important to notice here that although momentum is conserved, energy is not necessarily conserved. Particles will deform on impact and energy can be lost in another form such as heat or sound. Inelastic collisions disperse energy during a

collision. A perfectly inelastic collision implies that the particles do not separate after impact. So if $\vec{v}_{a1} = \vec{v}_{b1} = \vec{v}_1$ then this can be substituted into Eq. (10) to give

$$m_a \vec{v}_{a0} + m_b \vec{v}_{b0} = (m_a + m_b) \vec{v}_1$$

...Eq. (11)

For other inelastic collisions where the particles do separate from each other, energy is lost. The resulting velocities cannot be calculated because the system is indeterminate. Another equation can be obtained with the coefficient of restitution explored below.

In perfectly elastic collisions, all energy is conserved and none is lost due to the collision of the particles. The following video demonstrates a simple 2-pendulum collision similar to Newton's cradle.
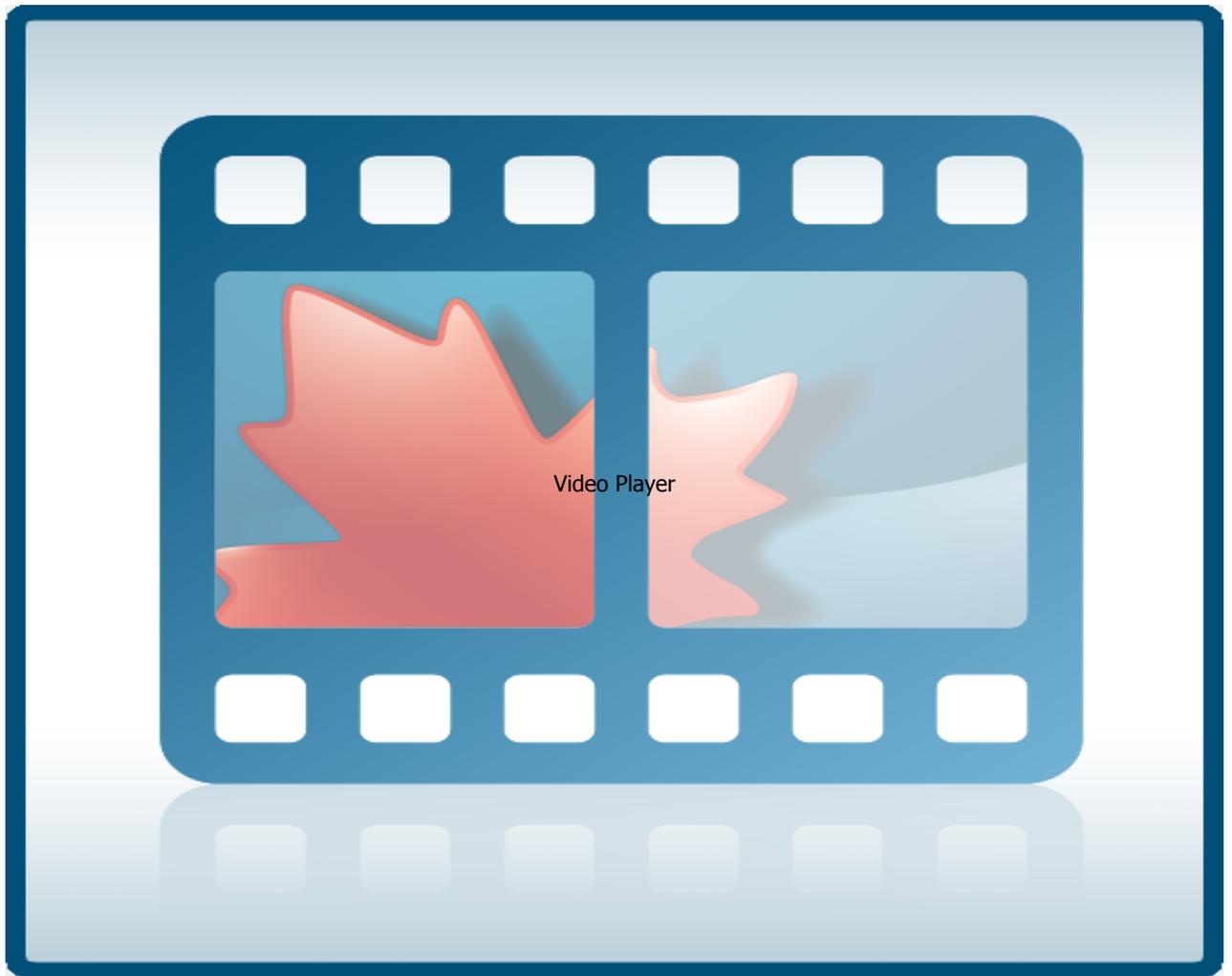
Video 1: Demonstration of momentum

Notice how the balls are stationary after they have transferred momenta. Each ball returns to the original drop height at the peak of each swing because no energy is lost. This is an example of an elastic collision under direct central impact. A larger model of Newton's cradle can be found on the MapleSim Model Gallery here.

Coefficient of Restitution

Particles can deform during a collision. As the particles deform, they approach an equal velocity. This is known as the period of deformation. Following this, the particles will either remain deformed, or return to their original shape, referred to as the period of restitution. These are demonstrated below.

a) Period of deformation

b) Period of restitution
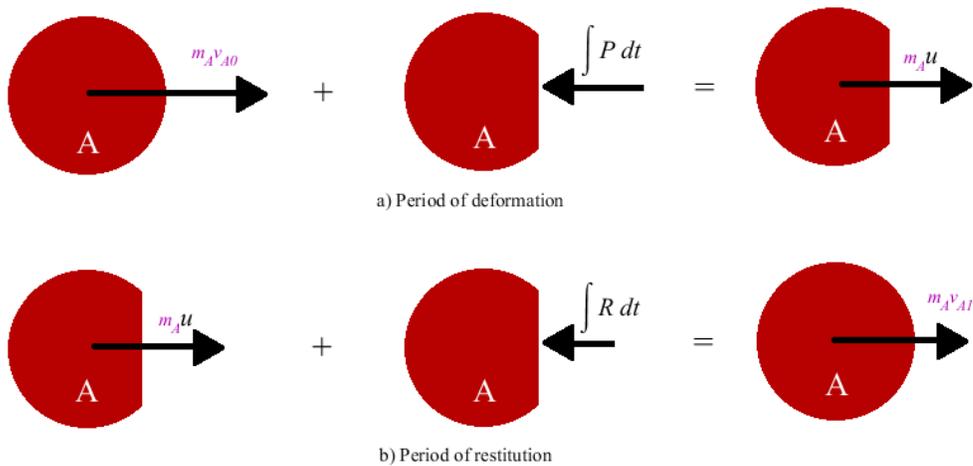
Figure 4: Period of deformation and restitution

Here, the ball A is rolling at velocity v_{A0} into another ball. From Eq. (5) the movement of ball A under impulsive motion can be examined in two parts. Figure 4 a) displays the period of deformation that the ball undergoes while it is coming into contact with the other ball. Both of the balls travel at the same velocity u at this point. The ball then enters the period of restitution, figure 4 b), where a restoring force $\int R\,dt$ will fix the deformation in A, after which the ball will be travelling at velocity v_{A1}. In practice, the restoring force R will always be less than or equal to the force of deformation P.

The **coefficient of restitution** is a ratio of the magnitudes of impulse between the period of restoration to the period of deformation.

$$e = \frac{\int R\,dt}{\int P\,dt}$$

... Eq. (12)

Solve for $\int R\,dt$ and $\int P\,dt$ from Eq.(5) and substitute into Eq. (12).

$$e = \frac{u - v_{A1}}{v_{A0} - u}$$

... Eq (13)

In an impact with another ball B, the forces R and P will act in the opposite directions. A similar analysis reveals

$$e = \frac{v_{B1} - u}{u - v_{B0}}$$

<div align="right">... Eq. (14)</div>

The two ratios for *e* are equal. Adding the numerators and denominators respectively will render the same solution.

$$e = \frac{v_{B1} - u + u - v_{A1}}{v_{A0} - u + u - v_{B0}}$$

<div align="right">... Eq. (15)</div>

This can be further simplified to

$$v_{B1} - v_{A1} = e \cdot \left(v_{A0} - v_{B0} \right)$$

<div align="right">... Eq. (16)</div>

This is the coefficient of restitution. It is especially useful in determining the resulting velocities of inelastic collisions. The characteristics of a collision can be examined with this ratio. A coefficient of restitution of 1 describes an elastic collision and a value of 0 results in a perfectly inelastic collision where each particle travels at the same speed.
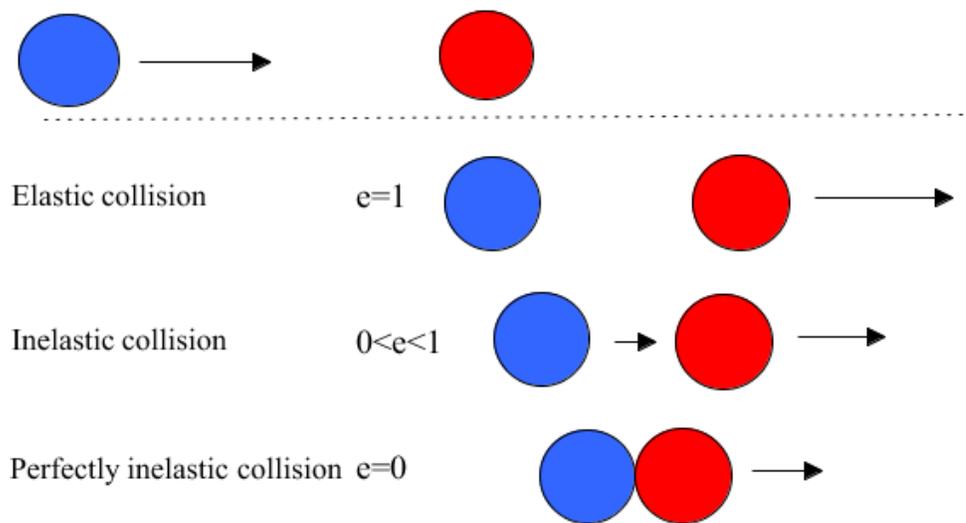
Figure 5: Varying the coefficient of restitution

Figure 5 displays the effect of changing the coefficient of restitution in a collision.

Experiment with the slider to change the coefficient of restitution for the impact in the animation below.

Coefficient of restitution, e =
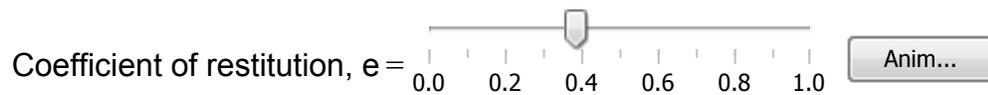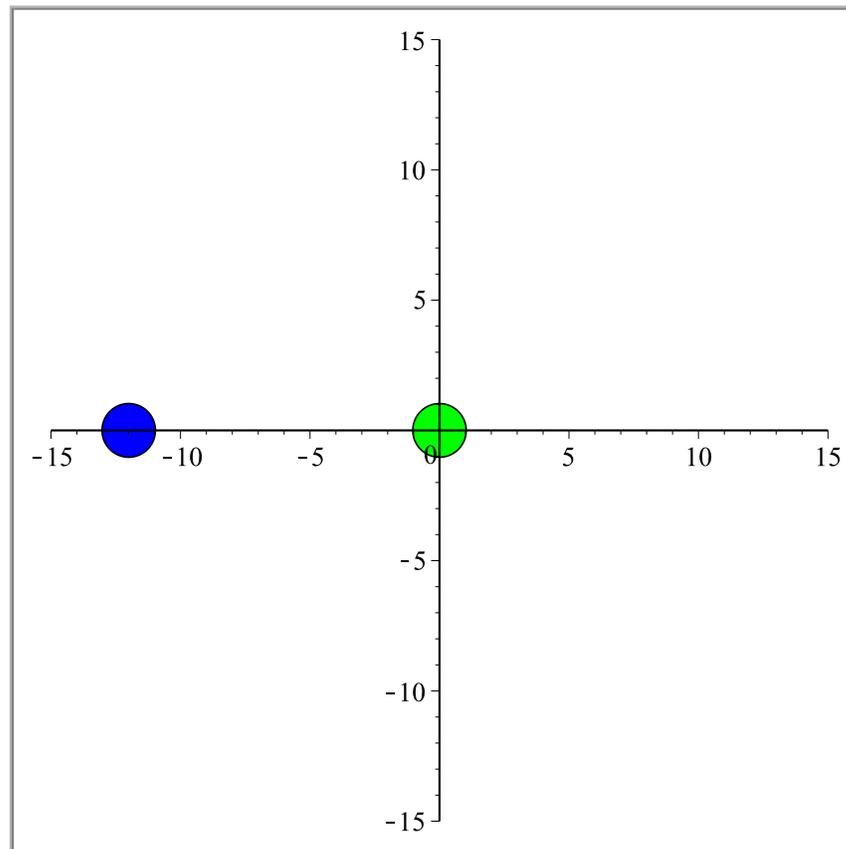
Animation 1: The coefficient of restitution in collisions

Examples with MapleSim

Example 1: Ball and Pendulum

Problem Statement: Ball B is hanging from an inextensible cord. An identical ball

A is released from rest when it touches the cord and aquires a velocity v_0 just before striking ball B. Assuming a perfectly elastic impact (e=1) and no friction, determine the velocity of each ball immediately after impact.
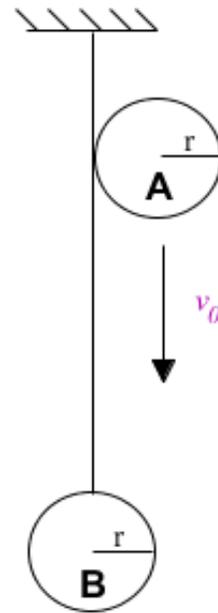
Figure 6: Ball and Pendulum

▼ Analytical Solution

restart;

Data:

$m_A = m_B = m$:

$v_{A0} := v_0$:

$v_{B0} := 0$:

$e := 1$:

Solution:

Ball B is constrained to move in a circle, so the direction is known to be horizontal immediately after impact. The magnitude of B and the direction and magnitude of A remain unknown. The angle that ball A strikes ball B can be examined as well. At a horizontal distance r from the cord, Ball A hits Ball B at a distance 2r from the center of each ball.
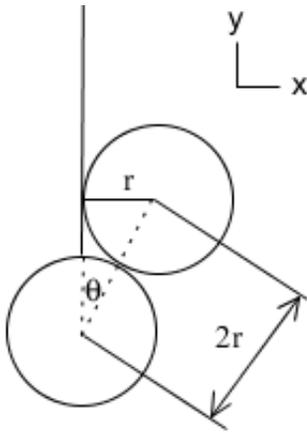
Figure 7: Ball and pendulum at collision

The angle that ball A strikes ball B,

$$\sin(\theta) = \frac{r}{2 \cdot r} = 0.5 :$$

The inverse sin solves for an angle of 30 degrees.

$$\theta = \frac{\pi}{6} :$$

This angle is used in the impulse-momentum equation for ball A.

$$m \cdot v_0 + F_A \cdot \Delta t = m \cdot v_{A1} :$$

The impulsive forces will be zero in this case because there is no energy lost (e=1) during the collision. In the tangent-normal frame of reference with respect to the point of impact, the momentum of A in the tangential direction can be expressed in the impulse-momentum equation.

$$m \cdot v_0 \cdot \sin(\theta) + 0 = m \cdot v_{A1_t} :$$
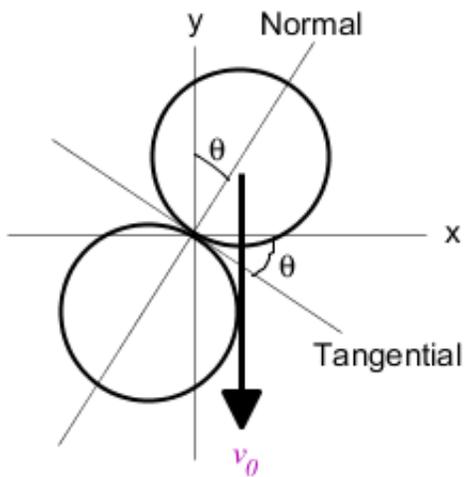
Figure 8: Transverse frame of reference

solving for the final velocity yields

$$v_{A1_t} = 0.5 \cdot v_0$$

Examining the collision in the x-direction will help solve for the normal velocity v_{A1_n} of ball A. For balls A and B,

$$0 = m \cdot v_{A1} + m \cdot v_{B1} :$$
$$0 = m \cdot v_{A1_t} \cdot \cos(\theta) - m \cdot v_{A1_n} \cdot \sin(\theta) - m \cdot v_{B1}$$

These components are expressed on the x-axis. Solving for the normal plane, the velocity of A becomes

$$v_{A1_n} = v_0 \cdot \cos(\theta) - 2 \cdot v_{B1}$$

Using the coefficient of restitution (with e=1) formula, the relative velocities expressed along the normal line of impact is

$$v_{B1} - v_{A1} = e \cdot (v_{a0} - v_{B0})$$

$$v_{BI}\cdot\sin(\theta) - v_{AI_n} = v_0\cdot\cos(\theta) - 0$$

Manipulate the equation to solve for the velocity of ball A,

$$v_{AI_n} = v_0\cdot\cos(\theta) - v_{BI}\cdot\sin(\theta)$$

solving this equation and the previous equation for v_{AI_n} the velocities of both ball A

and B can be expressed in terms of v_0.

$$v_{AI_n} = -0.520\cdot v_0$$

$$v_{BI} = 0.693\cdot v_0$$
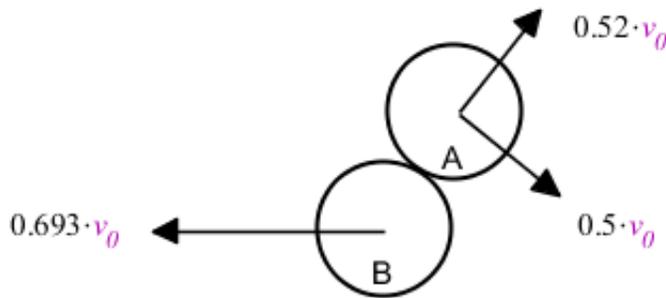
Figure 9: velocities of the balls

The velocity of ball A must be transformed back into the x-y plane. The magnitude of A is found from adding the velocity vectors in the tangential and normal planes using Pythagorean theorum,

$$|v_{AI}| := \sqrt{\left(-0.520\cdot v_0\right)^2 + \left(0.5\cdot v_0\right)^2} = 0.7213875519\cdot v_0$$

The angle that ball A will bounce away at is found through trigonometry with the tangent-normal vectors.

$$\beta := \arctan\left(\frac{0.5\cdot v_0}{-0.5196\cdot v_0}\right) = 46.1°$$

The angle above the x-axis α is

$$\beta - \theta = \alpha$$

$$\alpha = 46.1 - 30.0 = 16.1°$$

Therefore, after the balls collide, ball A leaves at a velocity of $0.721 \cdot v_0$ at an angle of 16.1° with respect to the x-axis. Ball B travels horizontally to the left with a velocity of $0.693 \cdot v_0$.

MapleSim Solution

Step 1: Insert Components

Drag the following components into a new workspace.
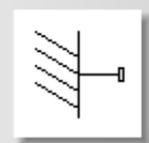
Component	Location
Fixed Frame	Multibody > Bodies and Frames
Rigid Body (2 required)	Multibody > Bodies and Frames
Rigid Body Frame	Multibody > Bodies and Frames
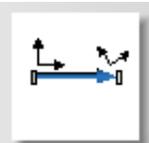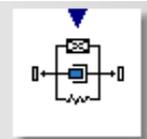
Translational Spring Damper Actuator	Multibody > Forces and Moments
Revolute	Multibody > Joints and Motions
Planar	Multibody > Joints and Motions
Spherical Geometry (2 required)	Multibody > Visualization
Vector Norm (2 required)	Signal Blocks > Mathematical > Operators
Less Equal Threshold	Signal Blocks > Relational
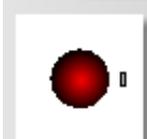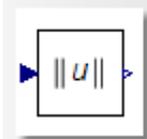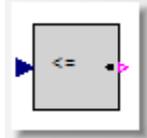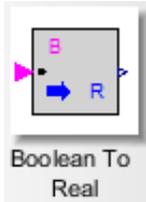
Boolean To Real	Signal Blocks > Signal Converters
Relative Translation	Multibody > Sensors
Absolute Translation	Multibody > Sensors
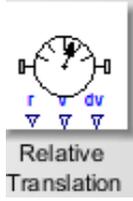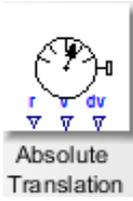
Step 2: Connect the components.

Connect the components as shown in the diagram below.
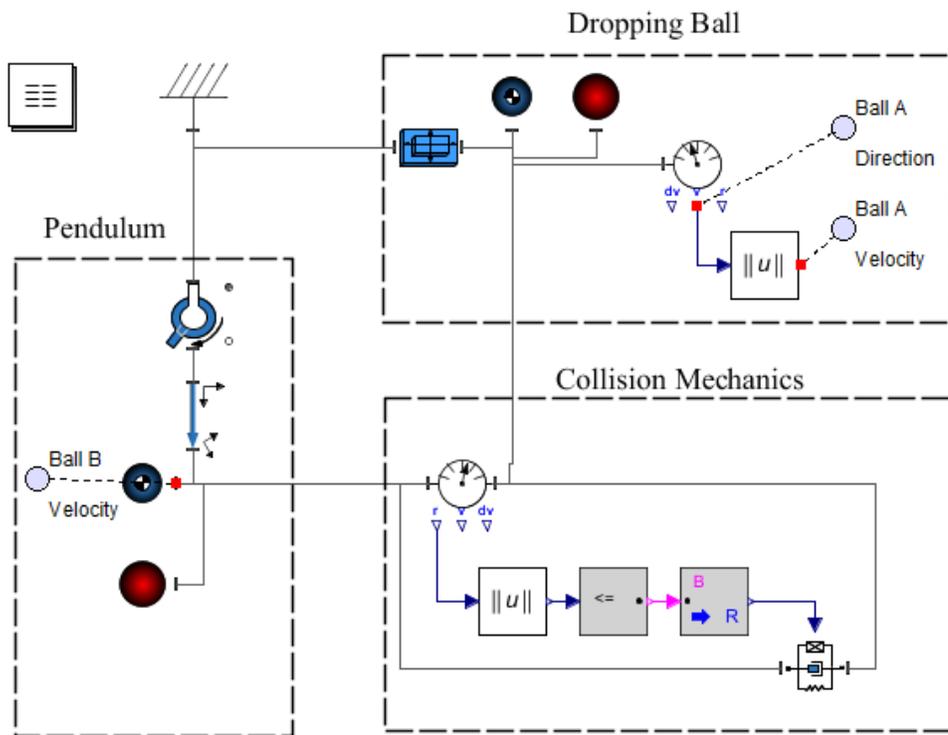
Figure 10: Model diagram

Step 3: Set up Ball A

1. In the main workspace, select the **parameters** ( ) icon and in the first table and under 'Name', type **r** and set the 'Default Value' to **0.5**. This will be the radius of the spheres A and B.

2. Select the **Sphere Geometry** block and on the 'inspector' tab, set the *radius* to **r**.

3. Select the **Rigid Body** block and on the 'inspector' tab, set the initial conditions $IC_{r, v}$ parameter to **Strictly Enforce**, and set the $\overline{r}_0$ parameter to **[r, 4, 0]**.

3. Connect a **probe** ( ) to the output of the **Vector Norm** block to measure the magnitude of velocity of ball A. Change the name of the probe to your liking.

4. Connect a **probe** ( ) to the **v** output of the **Absolute Translation Sensor** block and on the 'inspector' tab, check the **1** and **2** boxes to measure the x and y direction of travel.

Step 4: Set up Ball B

1. Select the **Sphere Geometry** block and on the 'inspector' tab, set the *radius* to **r**.

2. Select the **Rigid Body Frame** block and on the 'inspector' tab, set the $\bar{r}_{XYZ}$ parameter to **[0, -3.5, 0]**.

3. Select the **Fixed Frame** block and on the 'inspector' tab, set the $\bar{r}_{XYZ}$ parameter to **[0, 4, 0]**.

4. Connect a **probe** ( ) to the **Rigid Body** block and on the 'inspector' tab under 'Speed', check the **1** and **2** boxes to measure the x and y velocities, respectively.

Step 5: Set up Collision Mechanics

1. Ensure that you have connected Ball B's **Rigid Body** to the left side of the **Relative Translation Sensor** and **Translational Spring Damper Actuator** blocks, and connect Ball A's **Rigid Body** to the right sides.

2. Select the **Less Equal Threshold** block and on the 'inspector' tab, set the *threshold* parameter to **2*r**. This will send a 'true' signal when the balls are close enough to touch each other.

3. Select the **Boolean To Real** block and on the 'inspector' tab, set the *Real True* parameter to a high number like **10 000**. This is the magnitude of the force that acts on the balls as they touch. This is a perfectly elastic collision so the force should be very large as the balls hit.

All other parameters should be left at their default values.

Step 6: Run Simulation

1. Run the simulation.

2. Change the value of parameter r to see if it affects your results.

3. Obtain values from the generated graphs. To solve in terms of v_0 like the analytical solution did, divide the velocities by the value of the velocity of ball A before the collision.

▼ **Results**

The simulation generates the following results.
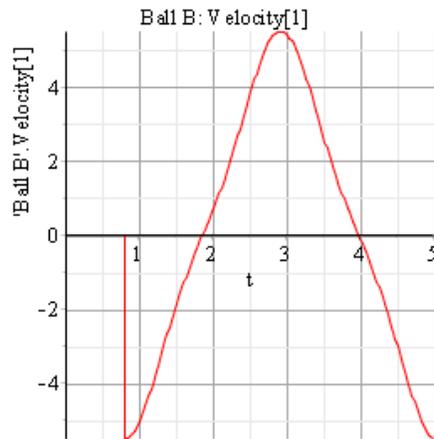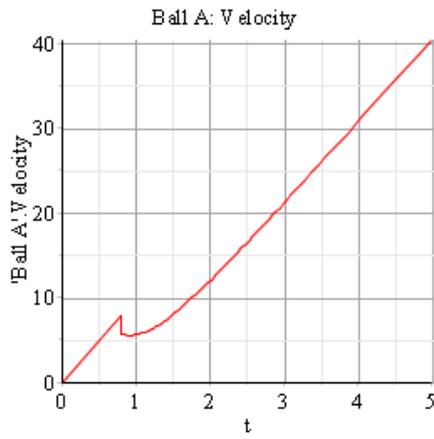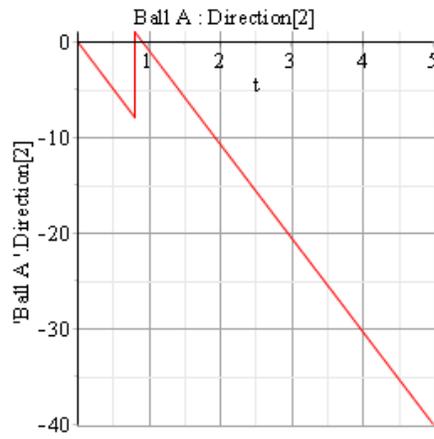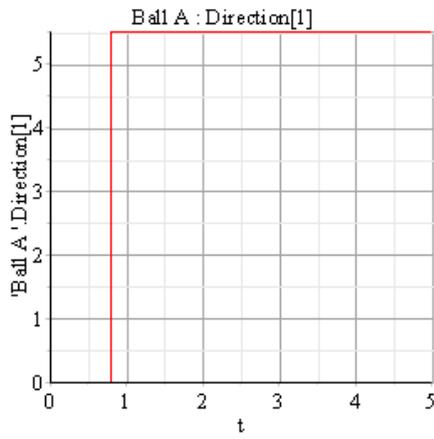
Figure 11: Simulation results

The following video demonstrates the model.
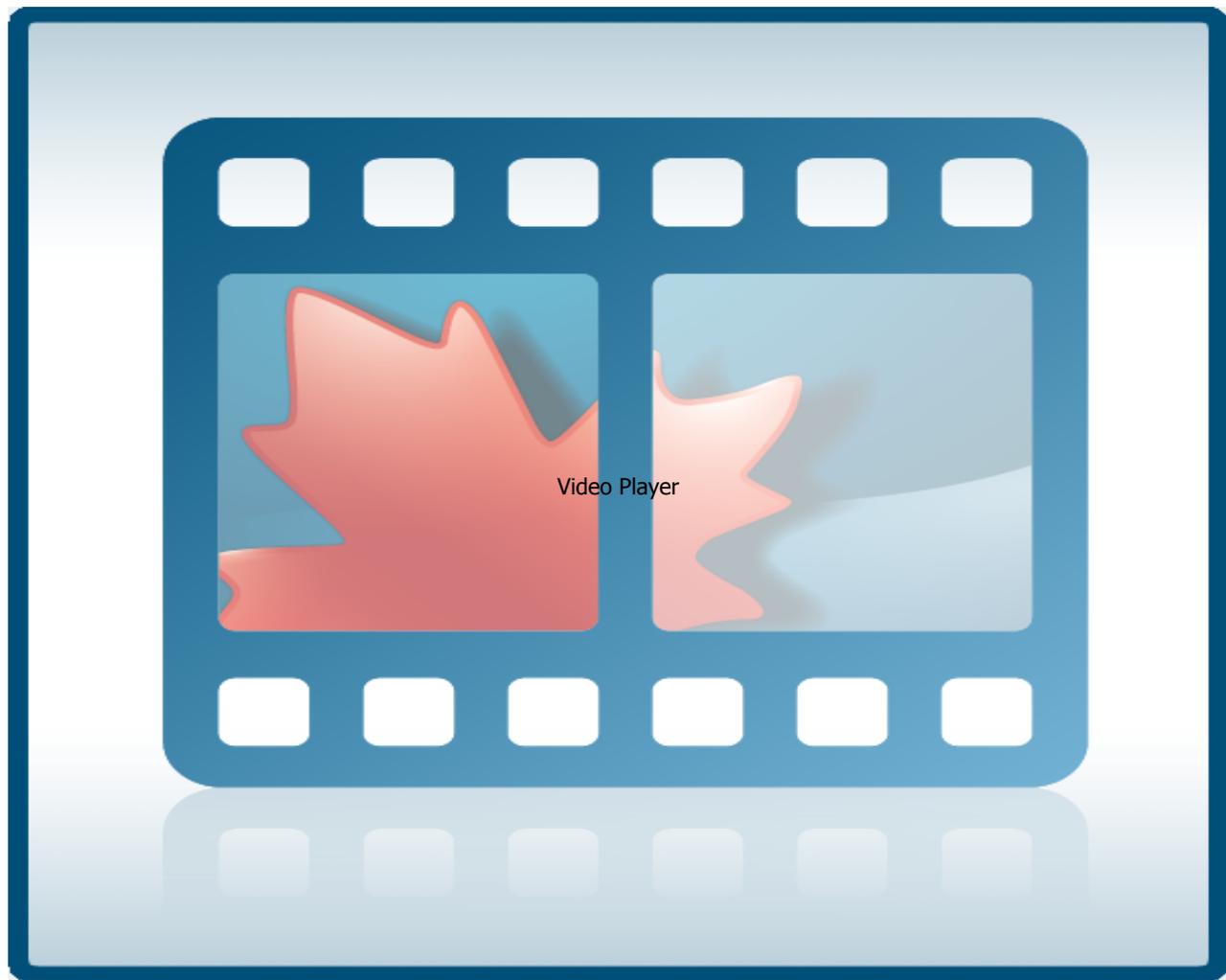
Example 2: Carrier Collision

Problem Statement: An employee pushes an empty 40-kg luggage carrier *A* with a velocity of 5 m/s into an identical second cart *B* containing a 15-kg suitcase equipped with rollers. The impact causes the suitcase to roll into the left wall of carrier *B*. Knowing that the coefficient of restitution is 0.8 between the carriers and 0.3 between the suitcase and carrier *B*, determine the velocity of luggage carrier B after the suitcase hits its wall.

v_{A0}
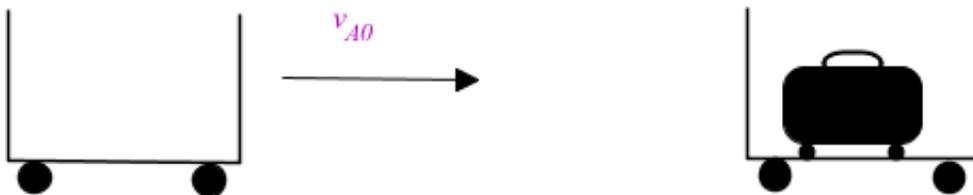
Figure 12: Carrier collision

Analytical Solution

restart;

Data:

$m_A := 40 \llbracket kg \rrbracket$:

$m_B := 40 \llbracket kg \rrbracket$:

$m_S := 15 \llbracket kg \rrbracket$:

$v_{A0} := 5 \dfrac{\llbracket m \rrbracket}{\llbracket s \rrbracket}$:

$v_{B0} := 0 \dfrac{\llbracket m \rrbracket}{\llbracket s \rrbracket}$:

$v_{S1} := 0 \dfrac{\llbracket m \rrbracket}{\llbracket s \rrbracket}$:

$e_C := 0.8$:

$e_S := 0.3$:

Solution:

Begin by finding the velocity of carrier *B* after its impact with *A*. The collision equation between them is

$eq1 := m_A \cdot v_{A0} + m_B \cdot v_{B0} = m_A \cdot v_{A1} + m_B \cdot v_{B1}$:

To obtain a second equation for our two unknowns, use the coefficient of restitution formula.

$eq2 := v_{B1} - v_{A1} = e_C \cdot (v_{A0} - v_{B0})$:

Isolate carrier *A*'s final velocity v_{A1}, substitute into the momentum equation and then solve for v_{B1}.

$solve(\{eq1, eq2\}, \{v_{A1}, v_{B1}\}) = \left\{ v_{A1} = \dfrac{0.5000000000 \; \llbracket m \rrbracket}{\llbracket s \rrbracket}, v_{B1} = \dfrac{4.500000000 \; \llbracket m \rrbracket}{\llbracket s \rrbracket} \right\}$

So vB1 = 4.5 m/s. Repeat this for the second collision between carrier B and the suitcase.

$v_{B1} := 4.5 \dfrac{\llbracket m \rrbracket}{\llbracket s \rrbracket}$:

$eq3 := m_B \cdot v_{B1} + m_S \cdot v_{S1} = m_B \cdot v_{B2} + m_S \cdot v_{S2}:$

Again, use the coefficient of restitution formula.

$eq4 := v_{S2} - v_{B2} = e_S \cdot (v_{B1} - v_{S1}):$

We want to find the final velocity of B, so isolate for the suitcase velocity v_{S2} and substitute into the momentum equation to obtain,

$$solve(\{eq3, eq4\}, \{v_{B2}, v_{S2}\}) = \left\{ v_{B2} = \frac{2.904545455 \; [\![m]\!]}{[\![s]\!]}, \; v_{S2} = \frac{4.254545455 \; [\![m]\!]}{[\![s]\!]} \right\}$$

Therefore, carrier *B* will be travelling at 2.90 m/s after the suitcase hits the inside wall.

MapleSim Solution

Step 1: Insert Components

Drag the following components into a new workspace.
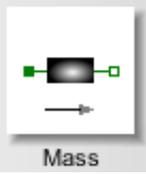
Component	Location
Mass (3 required)	1D Mechanical > Translational > Common
Translational Position (3 required)	1D Mechanical > Translational > Motion Drivers
Translational Relative Position Sensor	1D Mechanical > Translational > Sensors
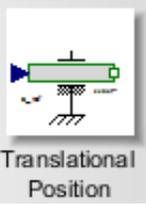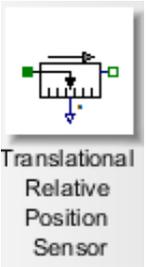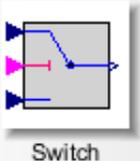
(2 required)	
Switch (4 required)	Signal Blocks > Boolean
Integrator (3 required)	Signal Blocks > Mathematical > Functions
Less Equal Threshold (2 required)	Signal Blocks > Relational
Hysteresis (2 required)	Signal Blocks > Signal Converters
Boolean To Real (2 required)	Signal Blocks > Signal Converters
Constant (3 required)	Signal Blocks > Sources > Real

Step 2: Connect the components.

Connect the components as shown in the diagram below.
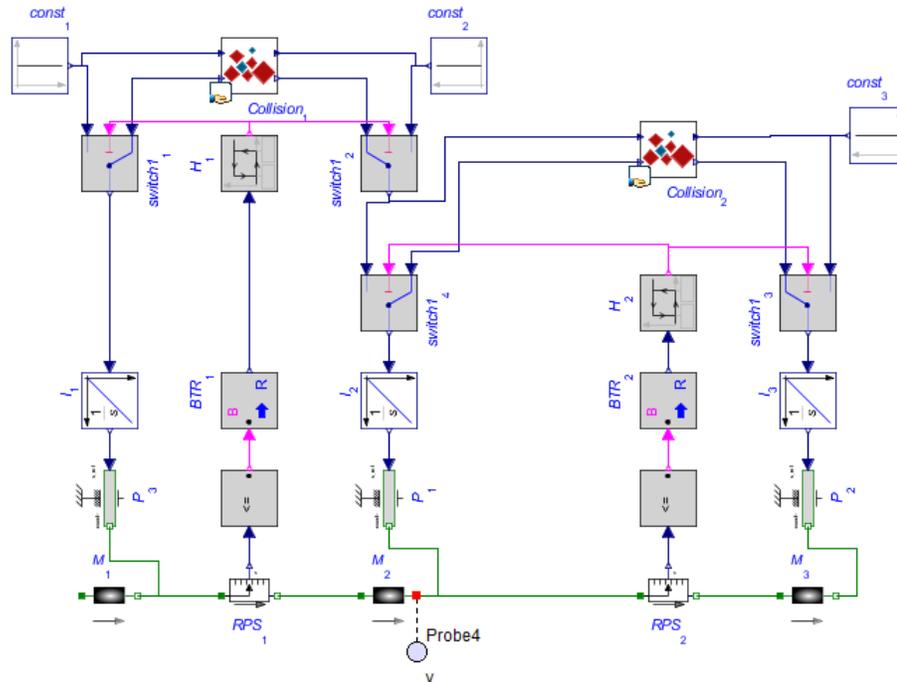
Figure 13: Model diagram

The 'Collision' blocks (red and blue diamonds) in the above image are custom components and will be made in the next step.

Step 3: Create custom component

1. From the file menu, click 'View', then select 'Create Attachment'. Choose 'Custom Component' and Maple will open with a new template.

2. Under the 'Component Equations' sub-heading, replace the contents of **eq** (3.1) with our collision equations, spearated by commas. The line should read,

$$eq := \left[m_J \cdot v_{J0}(t) + m_K \cdot v_{K0}(t) = m_J \cdot v_{J1}(t) + m_K \cdot v_{K1}(t), \; v_{K1}(t) - v_{J1}(t) = e \cdot \left(v_{J0}(t) \right. \right.$$

$$-v_{K0}(t))\bigr]$$

3. Make sure that you hit enter after completing each line. set the **params** value directly below to,

$$\text{params} := \bigl[m_J = 1,\, m_K = 1,\, e = 1\bigr]$$

4. Clear the contents of the **initialconditions** variable and hit enter.

5. Under the 'Component Ports' sub-heading, click the **Clear All Ports** button. Click the **Add Ports** button and add 4 ports to the component. Position each where you would like them on the block.

6. For each port, select the port and under 'Port Type', choose **Signal Input** and select either v_{J0} or v_{K0}. For the remaining two ports select them and under 'Port Type', choose **Signal Output** and choose either v_{J1} or v_{K1}. Each velocity should now be mapped to a port.

7. Under the 'Component Generation' sub-heading, click **Generate Component** to add it to your MapleSim model. The component can be found in the 'Library' tab under 'Definitions > Components'.

Step 4: Set Up Initial Conditions

1. Create a new **parameters** 📄 block and fill it with the following data:

Name	Type		Default Value	Default Units	Description
v_{A0}	Real	▼	5		Initial velocity of cart A
v_{B0}	Real	▼	0		Initial velocity of cart B
v_{S0}	Real	▼	0		Initial velocity of suitcase
m_A	Real	▼	40		Mass of cart A
m_B	Real	▼	40		Mass of cart B
m_S	Real	▼	15		Mass of suitcase
e_C	Real	▼	0.8		Coefficient of Restitution between carts
e_S	Real	▼	0.3		Coefficient of Restitution between suitcases

Figure 14: Parameters of the model

Step 5: Set Up Variable in Blocks

1. Select the appropriate **Mass** blocks and on the 'Inspector' tab, set the m parameter to m_A, m_B, and m_S. Also, set the v_0 initial conditions to v_{A0}, v_{B0}, and v_{S0}, where appropriate.

2. For each **Translational Position** block, select the 'exact' box on the 'Inspector' tab.

3. Select the **Integrator** block connected to mass B and S and on the 'Inspector' tab, set the y_0 parameter to 5 and 7, respectively. These values are not necessary, they are used to aid understanding.

4. For each **Hysteresis** block, set the u_{high} parameter to 0.5.

5. Set the appropriate **Constant** signal blocks to v_{A0}, v_{B0}, and v_{S0}.

6. Select a **custom component**. On the 'Inspector' tab, set the m_J and m_K parameters to m_A and m_B, respectively. Also set the e parameter to e_C for the carrier-on-carrier collision.

7. Select the other **custom component**. On the 'Inspector' tab, set the m_J and m_K parameters to m_B and m_S, respectively. Also set the e parameter to e_S for the suitcase-on-carrier collision.

Step 6: Connect probes

1. Connect a probe to flange b of the mass m_B, and on the 'Inspector' tab, check the 'Speed' measurement.

Step 6: Run Simulation

Run the simulation. Obtain the velocity of carrier B from the generated plot.

Results

The simulation generates the following results.
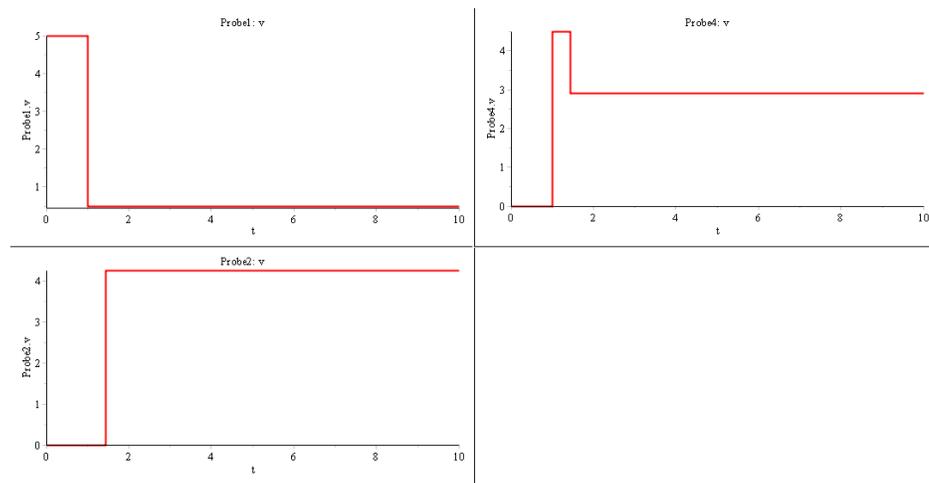
Figure 15: Simulation results

References:

http://www.maplesoft.com/applications/view.aspx?SID=3811&view=html

Beer et al. "Vector Mechanics for Engineers: Dynamics", 10th Edition. 1221 Avenue of the Americas, New York, NY, 2013, McGraw-Hill, Inc.